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On the Completeness of Multi-Variate Optimum Interpolation for Large-Scale Meteorological Analysis

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Abstract

The Baer-Tribbia non-linear modal initialization method implies that large-scale meteorological analyses should be confined to analysis of slow mode fields. An idealized multi-variate optimum interpolation analysis is shown to produce grid point results that contain only slow modes. Variational analysis with a slow mode constraint is therefore unnecessary.

1. The Consequences for Analysis of Non-Linear Modal Initialization

F. Baer and J. Tribbia (1977) have shown how initial conditions for a large-scale numerical model can be determined so that there is no "noise" in a forecast from this initial state. Their procedure is related to the Machenhauer method (1976) but is based on a formal expansion using the smallness of the ratio between Rossby mode frequencies and gravity mode frequencies. (These modes are the solutions to the linearized equations of the forecast model for perturbations on a resting basic state.) The procedure has been given a graphical description by the "slow manifold" diagram of Leith (1980). In figure 1, we consider the abscissa as equal to the total squared amplitudes of the slow modes contained in a complete three-dimensional meteorological field, and the ordinate as the corresponding measure of the "fast" mode components. (<u>Slow</u> and <u>fast</u> would normally be identical with <u>Rossby</u> and <u>gravity</u>, but are preferable names in that they recognize the freedom to choose the frequency separation criterion for best results.) The collection of all balanced states is represented by a curve in this highly compressed diagram, a curve to which Leith has given the name "slow manifold". This is because the Baer-Tribbia formalism assumes that in a balanced state, the fast mode components are not arbitrary, but are determined (i.e. forced) by the non-linear interaction of the slow modes. The atmosphere (and model) are assumed to be located on this manifold at all instants.

The Baer-Tribbia initialization process begins with a field containing only slow modes, such as would be obtained from a general analysis by subtracting all fast modes. The non-linear interactions of the slow modes produce tendencies, ($\partial u/\partial t$, etc.), which, in the first Baer approximation, are to be balanced by the linear tendencies of the unknown fast modes so that the total tendency of fast modes vanishes. This determines the first iterative

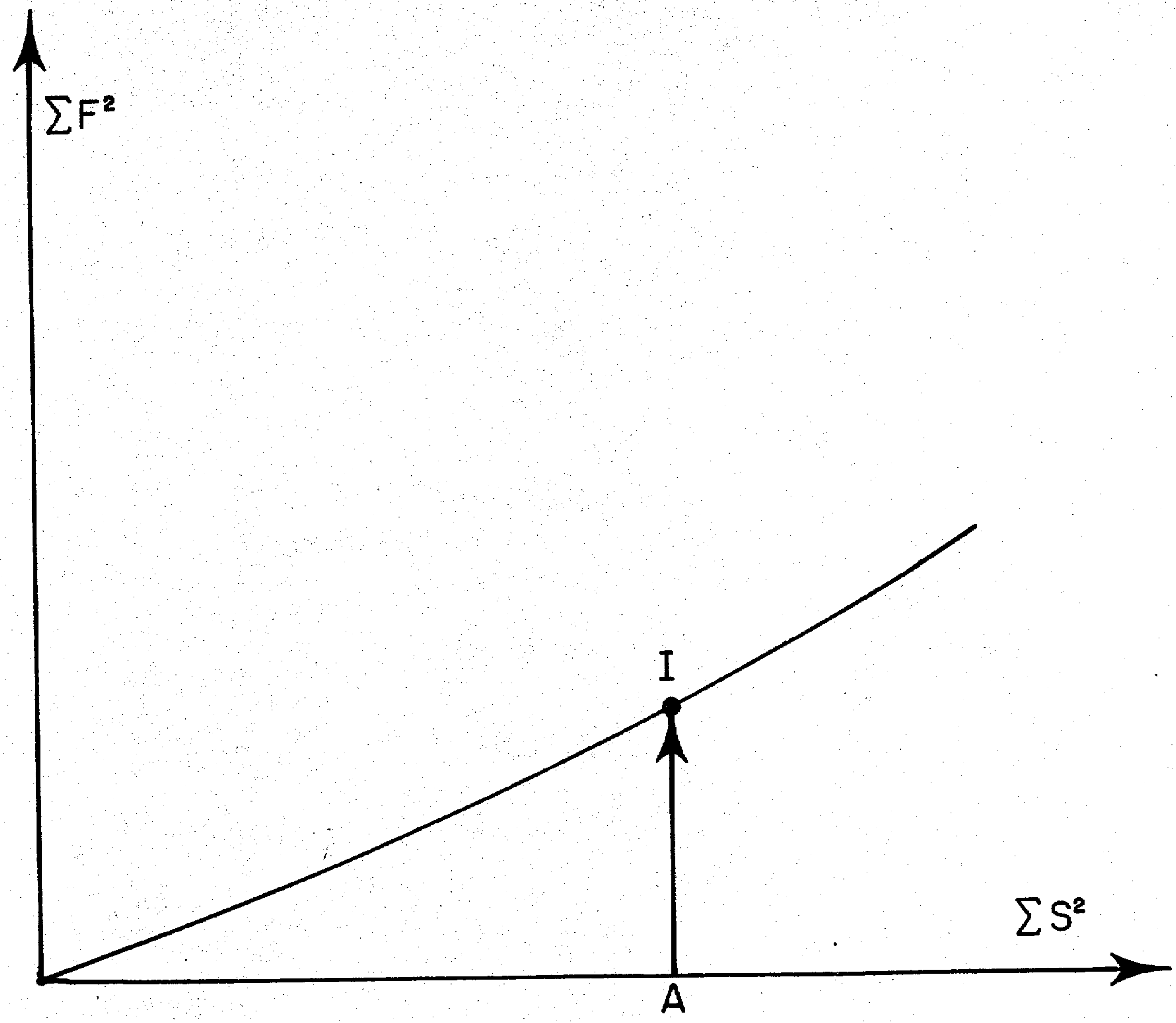

Fig. 1. The slow manifold of Leith. The Baer initialization process is indicated by the arrow from a slow mode analysis at A to the balanced state at I.

solution for the fast mode amplitudes.

The process for further iterations is well-defined, and the only mathematical problems are those of convergence of the iterations and the increasing computation associated with each iteration (Ballish, 1980). A meteorological problem also arises in that certain slow motions of importance involve a balance between strong release of latent heat and the vertical motions associated with gravity wave modes. Temperton (1980, p. 183) has suggested that the slow manifold must therefore be displaced upward in Leith's diagram as a "non-adiabatic" slow manifold to recognize this exception to the basic assumption underlying the Baer process.

This exception will require special consideration (for example, specification of the temperature tendency from latent heat as a known quantity) and in this paper I assume that a satisfactory treatment of this can be achieved. It is then possible to recognize the dependent character of the fast mode components in an initial field and draw the following logical consequences of the Baer initialization process:

I. <u>The purpose of large-scale meteorological analysis is to obtain the most accurate possible depiction of the slow mode fields</u>.

Two further consequences result immediately.

II. <u>Observations used in this slow mode analysis must be corrected for the fast mode components that they contain</u>.

III. <u>Any statistical-dynamical guidance used in analyzing slow mode fields must be based on the kinematic properties of only slow mode fields, not on the properties of complete fields</u>.

In a recent paper (1982) I have shown how procedure II can be implemented, and the importance of doing so with respect to obtaining maximum accuracy of the analysis in data-rich areas. This demonstration was couched in terms of the strict constraint variational analysis method introduced by Y. Sasaki (1958). As a useful technique, this analysis method is far removed from operational practice, however, because it is designed to use input data located only at grid points, and becomes extremely complex as soon as one begins to allow for the existence of correlations between the input data errors.

The remainder of this paper proves a "theorem" that obviates the use of a variational analysis to enforce the constraint that the analysis is to result in an analysis of slow modes only. The theorem can be stated as follows.

<u>A multivariate optimum interpolation analysis will result in grid-point values containing only slow modes if three conditions are met:</u>

1. It is given a first guess containing only slow modes.
2. The first guess error covariances that it uses are for slow mode errors only, and are specified by a power spectrum of slow mode error.
3. All observations are used in the analysis for each grid-point variable.

The theorem does not address the accuracy of the "observations" with respect to statement II above, this point having been addressed in the previous paper.

The theorem is first proven for a simple 1 dimensional domain. Section 2 describes this computation space and analysis grid, together with the mode definitions. Elementary geostrophic relations are used to define the slow modes. The optimum interpolation method is described in section 3, followed in section 4 by the spectral definition of the slow mode first guess error covariance structure. The Sasaki variational method, based on a slow mode constraint, is described in section 5. Particular emphasis is given to the requirement to use orthogonal error vectors in this calculation. It is shown in this section that if the input grid-point data for the variational process contains no fast modes the output field will be identical to the input field. The variational process will therefore be unnecessary if its input grid-point data contains no fast modes. The actual proof that the idealized optimum interpolation analysis of sections 3-4 contains no fast modes is deferred to section 6.

Section 7 provides a generalization to a typical 3-dimensional numerical prediction model and some concluding comments. The ideas of optimum interpolation that were originally formulated by Gandin (1963) and Eliassen (1954), and implemented recently by Lorenc (1981), are capable of responding completely to the theoretically based needs of modern large-scale meteorological analysis.

2. Analysis region and modes.

We consider a periodic x-domain, cyclically repeating over the distance 2π. This is marked off by an even number K of grid points:

$$x_k = 2\pi(k-1)/K \ , \quad k = 1, K \ . \tag{2.1}$$

At each grid point an analyzed field will have a "height" variable H and a "velocity" variable V. The Fourier representation of H and V will be defined as

$$H = \sum_{n=0}^{N} A_n \cos nx + \sum_{n=1}^{N-1} B_n \sin nx,$$
$$V = \sum_{n=0}^{N} C_n \cos nx + \sum_{n=1}^{N-1} D_n \sin nx, \tag{2.2}$$

in which

$$N = K/2. \tag{2.3}$$

x in (2.2) is a continuous variable, but the coefficients are determined by the grid point values:

For n=0:

$$(A_0, C_0) = \frac{1}{K} \sum_{k=1}^{K} (H_k, V_k). \tag{2.4}$$

For n=1, N:

$$(A_n, C_n) = \frac{1}{N} \sum_{k=1}^{K} (H_k, V_k) \cos nx_k, \tag{2.5}$$

For n=1, N-1:

$$(B_n, D_n) = \frac{1}{N} \sum_{k=1}^{K} (H_k, V_k) \sin nx_k \tag{2.6}$$

This convention is equivalent to that used in spectral transform forecast models.

In defining modes we will use the continuous form of (2.2) so that $\partial \cos nx / \partial x = -n \sin nx$, etc. The principle we follow is to associate the "slow" modes with the geostrophic-like relation

Slow modes: $$V = \partial H/\partial y \, . \tag{2.7}$$

The remaining modes, to be called "fast modes", will be orthogonal to the slow modes. Each mode has an H component and a V component, as listed in Table 1. The normalization factor M_n appearing in that table is given by

$$M_n = \left(\frac{2}{1+n^2} \right)^{1/2} \tag{2.8}$$

Let H_α, V_α denote the H and V components of mode α. Then the modes have the orthonormality property that

$$\frac{1}{2\pi} \int_0^{2\pi} \left(H_\alpha H_\beta + V_\alpha V_\beta \right) dy = \begin{cases} 0, & \alpha \neq \beta . \\ 1, & \alpha = \beta . \end{cases} \tag{2.9}$$

For n=0, the slow mode and fast mode represent, respectively, the x-averaged H field and the x-averaged V field (i.e. the "meridional circulation"). For n=1, N-1 the fast modes satisfy $H = \partial V/\partial y$. The special choice of modes for n=N is a simple one to resolve the vanishing of $\sin N x_k$ and the absence of B_N and D_N.

Table 1. Definition of modes. The factor $M_n = [2/(1+n^2)]^{1/2}$.

Wave Number	Type	H component	V component
n=0:	Slow	1	0
n=0:	Fast	0	-1
n=1,N-1:	Slow (even)	$M_n \cos nx$	$-nM_n \sin nx$
	Slow (odd)	$M_n \sin nx$	$nM_n \cos nx$
n=1,N-1:	Fast (even)	$nM_n \cos nx$	$M_n \sin nx$
	Fast (odd)	$nM_n \sin nx$	$-M_n \cos nx$
n=N:	Fast (even)	$\sqrt{2} \cos Nx$	0
	Fast (odd)	0	$\sqrt{2} \cos Nx$

Let S and F denote the amplitude of a slow or fast mode that is contained in a general field of the type (2.2)-(2.6). Their values are related to the Fourier coefficients according to the scheme in Table 2.

Table 2. Relation of modal amplitudes S and F to Fourier coefficients.

Wave Number	
n=0:	$S_o = A_o$
	$F_o = -C_o$
n=1,N-1:	$S_n(\text{even}) = (1/2)M_n\ (A_n - nD_n)$
	$S_n(\text{odd}) = (1/2)M_n\ (B_n + nC_n)$
n=1,N-1:	$F_n(\text{even}) = (1/2)M_n\ (nA_n + D_n)$
	$F_n(\text{odd}) = (1/2)M_n\ (nB_n - C_n)$
n=N:	$F_N(\text{even}) = A_N/\ 2$
	$F_N(\text{odd}) = C_N/\ 2$

An immediate relation to variational methods can be noted (Daley, 1978). Suppose $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$ represent a "filtered" field containing only wave numbers n=1, N-1, that is to be derived from an original field A_n-D_n by eliminating all fast modes. $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$ according to the recipe of Table 2 will be given by

$$\tilde{C}_n = n\tilde{B}_n = n\,(B_n + n\,C_n)/(1+n^2),$$
$$\tilde{D}_n = -n\tilde{A}_n = n\,(-A_n + n\,D_n)/(1+n^2). \qquad (2.10)$$

This filtering process can be shown to be equivalent to the variational problem of minimizing

$$\int_0^{2\pi} \left[(\tilde{H} - H)^2 + (\tilde{V} - V)^2 \right] d\lambda \qquad (2.11)$$

subject to the constraint

$$\tilde{V} = \partial \tilde{H} / \partial \lambda \qquad (2.12)$$

inasmuch as the differential equation resulting from (2.11) - (2.12) has the form

$$\tilde{H} - \partial^2\tilde{H}/\partial x^2 = H - \partial V/\partial x,$$

(a result derived in 1958 by Sasaki.) Our concern with variational analysis, however, will be to generalize (2.11) to allow for spatially variable accuracies of the input data H and V in the integrand of (2.11). This is done in section 5.

H and V have the same physical dimensions. If these are thought of as velocities, H is equal to the perturbation geopotential in a three-dimensional mode expansion multiplied by $2\pi/fL$, where f is the Coriolis parameter and L is the physical distance corresponding to the 2π interval in x. The fast modes treated here for n=1, N-1 are equal to the modes obtained by adding together the eastward and westward moving gravity wave modes associated with each wave number. They contain geopotential and vorticity but have zero divergence.

3. Optimum interpolation analysis

This process ("OI") arrives at an analysis by combining information from a "first guess" forecast and from observations with respect to the error structure of both. The procedure has been described most recently by Lorenc (1981). For convenience in notation, let us define a grid-point vector Y_k of length 2K by setting

$$Y_k = H_k \ (k=1,K) \text{ and } Y_k = V_{k-K} \ (k=K+1,2K). \tag{3.1}$$

We also define an observation vector Z_p of length R that represents P height observations,

$$Z_p = H_p(obs) \quad , (p=1,P) \tag{3.2}$$

at location x_p (not at grid points), followed by R-P wind observations

$$Z_p = V_{p-P}(obs) \quad , (p=P+1,R) \tag{3.3}$$

at their locations x_p. Because of the cyclic periodicity, these identical observations are repeated in every 2π interval in x.

The OI analysis is calculated as a correction to the first guess $\tilde{Y}_k$:

$$Y_k(an) = \tilde{Y}_k + \sum_{p=1}^{R} \alpha_{kp} (Z_p - \tilde{Y}_p). \tag{3.4}$$

Let z_p and y_p denote <u>errors</u> in the observation Z_p and the first guess Y_p, and let an overbar denote an ensemble average. We shall assume to begin with that observational errors and first guess errors are uncorrelated; $\overline{z_p y_q} = 0$. (This assumption is corrected at the end of section 6.) Under these conditions, the coefficients α_{kp} to minimize the ensemble squared error in Y_k (an) are determined by the following 2K sets of R equations (k=1, 2K; q=1,R; p=1,R)

$$\sum_{p=1}^{R} \alpha_{kp} (\overline{z_p z_q} + \overline{y_p y_q}) = \overline{y_k y_q} \quad . \tag{3.5}$$

Thus if O_{pq} denotes the inverse of the symmetric RxR error covariance matrix $\overline{z_p z_q} + \overline{y_p y_q}$, we can write

$$\alpha_{kp} = \sum_{q=1}^{R} O_{pq} \overline{y_k y_q}$$

The change from the first guess to the OI analysis is

$$Y_k(an) - \tilde{Y}_k = \sum_{p=1}^{R} (z_p - y_p) \sum_{q=1}^{R} O_{pq} \overline{y_k y_q} \tag{3.7}$$

and is a function only of the first guess error, the observational errors, and the location of the observations.

4. First guess errors and covariances

These are needed not only at the analysis points x_k but also at the observation points x_q. To be precise about this it is necessary to follow the same convention that was used in the mode definitions and use the Fourier series representation (2.2) - (2.6) to generalize from grid points to intermediate values of x.

Let $(a, b, c, d)_n$ represent the Fourier coefficients for the first guess error in an individual realization in the ensemble. If we introduce the first guess error notation

$$h = \tilde{H} - H(\text{true}), \qquad (4.1)$$
$$v = \tilde{V} - V(\text{true}),$$

we may write, for each realization in the ensemble:

$$h(x) = a_0 + \sum_{n=1}^{N-1} a_n \cos nx + b_n \sin nx, \qquad (4.2)$$

and

$$v(x) = c_0 + \sum_{n=1}^{N-1} c_n \cos nx + d_n \sin nx \qquad (4.3)$$
$$= \sum_{n=1}^{N-1} n(b_n \cos nx - a_n \sin nx),$$

where the last step follows from (2.10) and the fact that the first guess errors are errors in slow modes only.

Let ξ and η denote two values of x. Equations (4.2) and (4.3) can then be used to derive the following expressions for the error covariances of the slow mode first guess fields.

$$\overline{h(\xi)h(\eta)} = HHA + \sum_{m=1}^{N-1} HHC_m \cos m\xi + HHS_m \sin m\xi, \qquad (4.4)$$

$$\overline{h(\xi)v(\eta)} = HVA + \sum_{m=1}^{N-1} HVC_m \cos m\xi + HVS_m \sin m\xi, \tag{4.5}$$

$$\overline{v(\xi)h(\eta)} = \sum_{m=1}^{N-1} m\left(HHS_m \cos m\xi - HHC_m \sin m\xi\right), \tag{4.6}$$

$$\overline{v(\xi)v(\eta)} = \sum_{m=1}^{N-1} m\left(HVS_m \cos m\xi - HVC_m \sin m\xi\right). \tag{4.7}$$

In these expressions the ξ dependence appears only through $\cos m\xi$ and $\sin m\xi$. The HHA - HVS terms depend only on η and (except for HHA and HVA) on m. They include the statistical information by means of covariances of the Fourier coefficients of the first guess error.

$$HHA = \overline{a_0^2} + \sum_{n=1}^{N-1} \left(\overline{a_0 a_n} \cos n\eta + \overline{a_0 b_n} \sin n\eta\right), \tag{4.8}$$

$$HVA = \sum_{n=1}^{N-1} n\left(\overline{a_0 b_n} \cos n\eta - \overline{a_0 a_n} \sin n\eta\right), \tag{4.9}$$

$$HHC_m = \overline{a_0 a_m} + \sum_{n=1}^{N-1} \left(\overline{a_m a_n} \cos n\eta + \overline{a_m b_n} \sin n\eta\right), \tag{4.10}$$

$$HHS_m = \overline{a_0 b_m} + \sum_{n=1}^{N-1} \left(\overline{b_m a_n} \cos n\eta + \overline{b_m b_n} \sin n\eta\right), \tag{4.11}$$

$$HVC_m = \sum_{n=1}^{N-1} n\left(\overline{a_m b_n} \cos n\eta - \overline{a_m a_n} \sin n\eta\right), \tag{4.12}$$

$$HVS_m = \sum_{n=1}^{N-1} n\left(\overline{b_m b_n} \cos n\eta - \overline{b_m a_n} \sin n\eta\right). \tag{4.13}$$

This representation is quite general, including even systematic geographical error represented by non-zero values of $\overline{a}_m$ and $\overline{b}_m$.

Formulas (4.4) - (4.13) will be used in section 6 to prove that the OI analysis (3.7) contains only slow modes.

5. Variational analysis

In order to prove the theorem that a multivariate optimum interpolation analysis of slow mode fields makes a variational analysis unnecessary, it is of course necessary to describe what type of variational analysis is meant. Suppose we have input values of the grid point variables H_k^o, V_k^o. (For our purpose H_k^o and V_k^o will be the OI analysis values.) We must also know the associated error covariance matrix E_{ij} in which i and j both take on the values 1 through 2K. To be specific, E_{ij} can be constructed according to the scheme used in (3.1) for the optimum interpolation: $E_{ij} = \overline{\delta Y_i^o \delta Y_j^o}$. Let δH^o and δV^o be the error in H^o and V^o. We set

$$
\begin{aligned}
i = 1,K;\ j = 1,K: &\quad E_{ij} = \overline{\delta H_i^o \delta H_j^o} \\
i = K+1,2K;\ j = 1,K: &\quad E_{ij} = \overline{\delta V_{i-K}^o \delta H_j^o} \\
i = 1,K;\ j = K+1,2K: &\quad E_{ij} = \overline{\delta H_i^o \delta V_{j-K}^o} \\
i = K+1,2K;\ j = K+1,2K: &\quad E_{ij} = \overline{\delta V_{i-K}^o \delta V_{j-K}^o}
\end{aligned}
\qquad (5.1)
$$

As a covariance matrix, E_{ij} will have 2K non-negative eigenvalues λ_k and each of these will have a 2K component eigenvector ξ_{ik} (i=1,2K) associated with it that is orthonormal to the other ξ_{il}.

The essence of the variational procedure is that introduced by Sasaki (1958). Using input values X_l^o, we seek to obtain analysed values X_l that will minimize a positive definite functional of the form

$$SAS = \sum_{l=1}^{L} W_l (X_l - X_l^o)^2, \quad W_l > 0 \qquad (5.2)$$

and also obey M linear constraints*

$$\sum_{l=1}^{L} R_{ml} X_l = 0 \qquad (m=1,M;\ M \le L). \qquad (5.3)$$

* Sasaki stated the constraint as a differential equation. As long as this differential equation constraint is linear, we may use Fourier series for our cyclic x region to express the constraint in the form (5.3). See also the discussion at the end of section 2.

The weights W_l in (5.2) must be positive for that expression to be positive definite. It can readily be shown (e.g. Phillips, 1982) that if the errors in the input data X_l^0 are independent - that is if $\overline{\delta X_l^0 \delta X_m^0} = 0$ when $l \neq m$ - then the optimum choice for W_l is the reciprocal of $\overline{(\delta X_l^0)^2}$:

$$W_l\,(opt) = 1 \,/\, \overline{(\delta X_l^0)^2}\,. \qquad (5.4)$$

In our case the input data X_l^0 is represented by the 2K values of H_k^0, V_k^0 from the multivariate optimum interpolation analysis. From (3.8) we have the following formula for the analysis error δY_k as a tool to compute $\overline{\delta Y_k \delta Y_l}$:

$$\delta Y_k = y_k + \sum_{p=1}^{R} (z_p - y_p) \sum_{q=1}^{R} O_{pq}\, \overline{y_k y_q} \qquad (5.5)$$

$\overline{\delta Y_k \delta Y_l} = E_{kl}$ will not vanish in general for $k \neq l$. The procedure by which we can recover the desired condition (5.4) in these circumstances is to transform the variables H and V by projecting them onto the eigenvectors of E_{ij}:

$$w_l^0 = \sum_{i=1}^{K} \xi_{il} H_i^0 + \sum_{i=1}^{K} \xi_{i+K\,l} V_i^0\,, \quad l=1,2K, \qquad (5.6)$$

Errors in w_l^0 and w_m^0 are now uncorrelated, and the squared error in w_l^0 is simply λ_l. Our positive definite functional becomes

$$SAS = \sum_{l=1}^{2K} \frac{(w_l - w_l^0)^2}{\lambda_l} = \text{minimum}\,. \qquad (5.7)$$

The linear constraint (5.3) must now be stated. Our constraint will be that <u>the output field from the variational step must contain only slow modes</u>. This can be stated explicitly as a relation on w_l by using the slow mode Fourier relation (2.10) as a constraint on the variational output grid values H_k and V_k:

$$
\begin{aligned}
H_k &= A_0 + \sum_{m=1}^{N-1} (A_m \cos m\psi_k + B_m \sin m\psi_k) \\
V_k &= \sum_{m=1}^{N-1} m\,(B_m \cos m\psi_k - A_m \sin m\psi_k).
\end{aligned}
\tag{5.8}
$$

We then project these onto $\xi_{i\ell}$ in the manner of (5.6) for substitution into (5.7) for w_ℓ. The resulting expression for SAS has 2N-1=K-1 undetermined parameters: A_0 through A_{N-1} and B_1 through B_{N-1}. K-1 linear equations for these parameters will result from setting the partial derivatives of SAS with respect to A_0, A , ..., B_{N-1} equal to zero.

Suppose however that in the error covariance matrix E_{ij} there is one linear combination of variables

$$
G = \sum_{i=1}^{K} (f_i H_i^0 + g_i V_i^0) \tag{5.9}
$$

that has no error. This will make one eigenvalue, λ_{2K} (say), have a value of zero. If there are 10 such error-free linear combinations, 10 of the eigenvalues will be zero. From the form of SAS in (5.7), the weights λ^{-1} for these w_ℓ will be infinite, instructing the variational analysis, so to speak, to leave these w_ℓ values equal to their input values w_ℓ^0.

In section 6 it will be shown that the multivariate OI analysis has the property that Y_k (anal) has no components in the K+1 fast modes described in section 2. The same condition will apply to the errors in the multivariate OI analysis fields H_k (anal) and V_k (anal). If these are the input to the variational problem, there will be K+1 linear combinations like (5.9) in E_{ij}. The upper limit L in the sum (5.7) is reduced from 2K to K-1. Since this is equal to the number of unknowns, the resulting variational problem has the trivial result

$$w_l = w_l^0 \quad , \quad l = 1, 2K \tag{5.10}$$

In other words, the variational problem has been given input data that exactly satisfies the constraint (5.8).

6. <u>Absence of fast mode components in the optimum interpolation analysis.</u>

Equation (3.7) shows that the analyzed field Y_k (an) from the idealized optimum interpolation is equal to the first guess value plus a correction term:

$$Y_k(an) = \tilde{Y}_k + \sum_{p=1}^{R} (z_p - y_p) \sum_{q=1}^{R} O_{pq} \, \overline{y_k y_q} \ . \tag{6.1}$$

z_p is the observation error at x_p , y_p and y_q are the first guess error at observing points p and q and y_k is the first guess error at the grid point corresponding to Y_k . Our fundamental assumption is that we are analyzing only slow mode fields, so that the correction term is a correction to the first guess slow mode fields. Thus, since the first guess $\tilde{Y}_k$ contains only slow modes, we can prove that Y_k (an) contains only slow modes by demonstrating that the correction term in (6.1) has no projection onto any of the K+1 fast modes defined in section 2.

We consider first the case of fast modes for n=1,N-1. From Table 2 and (2.5) - (2.6) we can write the following formulas for these fast mode amplitudes F_n:

$$N\left[2(1+n^2)\right]^{1/2} F_n(\text{even}) = \sum_{k=1}^{K} n H_k \cos n x_k + V_k \sin n x_k , \tag{6.2}$$

$$N\left[2(1+n^2)\right]^{1/2} F_n(\text{odd}) = \sum_{k=1}^{K} n H_k \sin n x_k - V_k \cos n x_k . \tag{6.3}$$

The reasoning in the preceding paragraph shows that these will be zero if we also obtain zero when H_k and V_k on the rights side of (6.2) - (6.3) are replaced by the corresponding parts for H_k and V_k of the correction terms on the right side of (6.1). Since the k-sums of (6.2) - (6.3) can be brought inside the p,q sums of (6.1), it is in fact only necessary to show that sums of the type appearing on the right side of (6.2) and (6.3) vanish if H_k and V_k in those equations are replaced by the appropriate part of $\overline{\eta_k \eta_q}$.

$$\text{For } H_k: \quad \overline{h_k \eta_q} = \overline{h_k h_q} \ \underline{\text{and}} \ \overline{h_k v_q} \tag{6.4}$$

$$\text{For } V_k: \quad \overline{v_k \eta_q} = \overline{v_k h_q} \ \underline{\text{and}} \ \overline{v_k v_q}. \tag{6.5}$$

Thus, for F_n (even) to vanish, it must be shown that the k-sum (k=1,K) of

$$I_k = n \cos n x_k \, \overline{h_k h_q} + \sin n x_k \, \overline{v_k h_q}, \tag{6.6}$$

and the k-sum of

$$J_k = n \sin n x_k \, \overline{h_k v_q} - \cos n x_k \, \overline{v_k v_q} \tag{6.7}$$

both vanish. h_q in (6.6) and v_q in (6.7) carry information about the observation locations for H(obs) and V(obs). Clearly, the theorem, if it is to be useful, must not depend on the mixture or location of observations, and this is why both I_k and J_k must vanish, and do so for any value of x_q.

The expressions for $\overline{h_k h_q}$, etc., are given by (4.4)-(4.13), if ξ is interpreted as $x_k = 2\pi(k-1)/K$, and η is interpreted as x_q. The HHA - HVS terms do not depend on k. Summation over k=1,K of (4.4)-(4.7) after multiplication with $\sin m x_k$ and $\cos m x_k$ is simplified by the summation rules that hold when n and m range from 1,N-1. These are that the k-sums of $\cos m x_k$,

$\sin n x_k$, and $(\cos m x_k \sin m x_k)$ vanish, and that the sums of either $(\cos n x_k \cos m x_k)$ or $(\sin n x_k \sin m x_k)$ also vanish except for n=m, when their sums are equal to K/2=N. For example, the k (and m) summations of the first term in J_k gives

$$\sum_{k=1}^{K} n \cos n x_k \overline{h_k h_g} = N n \, HHC_n \tag{6.8}$$

The sum of the second term in I_k is found to be the negative of this, so that the k-sum of I_k is zero. Similarly the k-sum of the first term in J_k is

$$\sum_{k=1}^{K} n \sin n x_k \overline{h_k v_g} = N n \, HVS_n \tag{6.9}$$

The sum of the second J_k term is again the negative of this. This proves that F_n (even) vanishes for n=1,N-1. A similar manipulation will show that F_n (odd) is also zero.

F_o, according to Table 2 and equation (2.4), is proportional to the k-sum of V_k(an). From (3.8), this will vanish if the k-sum of $\overline{v_k v_g}$ and $\overline{v_k h_g}$ both vanish. This is assured by the linearity of (4.7) and (4.6) with respect to $\sin m\xi$ and $\cos m\xi$ (ξ is replaced by x_k). The vanishing of F_N (even) and F_N (odd) is made obvious from Table 2 by noting that A_N and C_N depend on the sum of terms proportional to $\cos N x_k$, and that wave number N is missing in the first guess slow mode error covariances given in (4.4) - (4.13).

This proves that the analysis produced by the optimum analysis system described in section 3 will contain only slow modes if (a) it is provided with a first guess containing only slow modes, and (b) it uses first guess error covariances based on a proper spectral representation of error covariances in those slow modes, and (c) every observation is used in the analysis for each grid point.

As a final step, it is possible to extend the analysis to include the situation where the errors ζ_q in observations are correlated with the first guess error $y = h$ or v. In section 2 this effect is introduced by adding the terms $-(\overline{\zeta_p y_q} + \overline{\zeta_q y_p})$ to the matrix multiplying α_{kp} in (3.5) and by subtracting the term $\overline{y_k \zeta_q}$ from the right side. If O_{pq} now represents the inverse of the new optimum interpolation matrix, the only corrections needed to the arguments of this section can be condensed into the requirement that the k-sums of both

$$L_k = -m\cos m\phi_k\, \overline{\zeta_q h_k} - \sin m\phi_k\, \overline{\zeta_q v_k},$$

$$M_k = -m\sin m\phi_k\, \overline{\zeta_q h_k} + \cos m\phi_k\, \overline{\zeta_q v_k} \qquad (6.10)$$

vanish. To investigate this, we first require that $\overline{\zeta_q y_k}$ can be written formally as

$$\overline{\zeta_q h_k} = \overline{a_0 \zeta_q} + \sum_{m=1}^{N-1} \left(\overline{a_m \zeta_q}\cos m\phi_k + \overline{b_m \zeta_q}\sin m\phi_k\right), \qquad (6.11)$$

$$\overline{\zeta_q v_k} = -\sum_{m=1}^{N-1} m\left(\overline{a_m \zeta_q}\sin m\phi_k - \overline{b_m \zeta_q}\cos m\phi_k\right) \qquad (6.12)$$

Terms such as $\overline{a_m \zeta_q}$ in these equations must be assigned a precise meaning for any realization, in order to use these equations to prove this extension of the theorem. Let the observation q represent a measurement of a variable Z (H or V in our model). The observation error ζ_q can be expressed as the sum of a random part ϵ_q and a term proportional to the true variable

$$\zeta_q = \epsilon_q + \rho_q Z_q(\text{true})$$

with a known value for ρ appropriate to this type of observation. Z_q (true) is also equal to Z_q (first guess) - y_q (first guess error in Z_q). Then

$$a_m \zeta_q = \frac{1}{N} \sum_{k=1}^{K} h_k \left\{\epsilon + \rho\left[Z(\text{f.guess}) - y\right]\right\}_q \cos m\phi_k$$

Thus for the observation locations that happen to occur in one realization, $\overline{a_m \mathfrak{z}_q}$ can be expressed in principle as a valid statistical estimate if one is willing to specify the covariances of the first guess (H or V) itself at point q with the first guess error h_k at point k. The knowledge that this is possible is sufficient to justify our use of the formal expressions (6.11) and (6.12).

For both L_k and M_k it is again found that the k-sum of their first and second terms cancel. The theorem also applies therefore to the situation of correlated observational errors and first guess errors.

7. Extension to Three Dimensions

This is a straight-forward matter because much of the notation of previous sections can be retained. The subscript k on the grid-point data vector Y_k in (3.1) is generalized first so as to denote the complete set of dynamic variables at all grid points. If $k = 1, K_0$ denotes the number of "components" in Y_k, in a typical numerical prediction model K_0 will be given by

$$K_0 = K_H \times (1 + 3K_V) \tag{7.1}$$

K_H and K_V denote the number of horizontal and vertical grid points, respectively. The factor 3 allows for 2 horizontal velocity components and a temperature at each grid point, while the added 1 represents a surface pressure field.

We must generalize the modal definitions of section 2. To this end we postulate the existence of a complete orthonormal set of eigenfunctions $\phi_\ell(\underset{\sim}{x})$, $\ell = 1, K_0$ in which the vector $\underset{\sim}{x}$ denotes not only the three continuous space variables but also the type of variable. These allow an arbitrary grid point field to be expressed as a sum of the eigenvectors

$$Y_k = Y(\underset{\sim}{x}_k) = \sum_{l=1}^{K_0} \Phi_l \phi_l(\underset{\sim}{x}_k). \tag{7.2}$$

The associated continuous field is given by the same expression without the subscript k. For convenience we can assume that the orthogonal eigenvectors are normalized with respect to a simple grid point sum

$$\sum_{k=1}^{K_0} \phi_l(\underset{\sim}{x}_k)\phi_m(\underset{\sim}{x}_k) = \begin{matrix} 0 & l \neq m \\ 1 & l = m \end{matrix} \tag{7.3}$$

The expansion coefficients Φ_l are then given by

$$\Phi_l = \sum_{k=1}^{K_0} Y_k \phi_l(\underset{\sim}{x}_k). \tag{7.4}$$

The notation of section 3 for the OI analysis is general. We will only need (3.7),

$$Y_k(an) = \tilde{Y}_k + \sum_{p=1}^{R} (z_p - y_p) \sum_{q=1}^{R} q_{pq} \overline{y_k y_q} \tag{7.5}$$

which gives the OI analyzed values as a function of the first guess $\tilde{Y}_k$ and the error of observation z_p and of the first guess errors y_p, y_q, y_k. We need now the first guess error covariances $\overline{y_k y_q}$. Let the set ϕ_l be divided into two distinct sets: a "slow" set $\psi_l = \phi_l$, $l = 1, L_0$ and a "fast" set $\chi_l = \phi_l$, $l = L_0+1, K_0$. The generalization of (4.2)-(4.3) is

$$y(\underset{\sim}{x}) = \sum_{l=1}^{L_0} \Psi_l \psi_l(\underset{\sim}{x}) \tag{7.6}$$

as a spectral representation of the slow mode error in an individual realization. We then find that

$$\overline{y_k y_q} = \overline{y(\underline{x}_k)\, y(\underline{x}_q)}$$

$$= \overline{\sum_{l=1}^{L_0} \Psi_l\, \psi_l(\underline{x}_k) \cdot \sum_{m=1}^{L_0} \Psi_m\, \psi_m(\underline{x}_q)}$$

$$= \sum_{l=1}^{L_0} \psi_l(\underline{x}_k) \sum_{m=1}^{L_0} \overline{\Psi_l \Psi_m}\, \psi_m(\underline{x}_q)$$

$$= \sum_{l=1}^{L_0} T_l(\underline{x}_q)\, \psi_l(\underline{x}_k) \tag{7.7}$$

This compact expression is the generalization of (4.4)-(4.13).

The projection of (7.5) onto a fast mode is obtained by multiplying (7.5) with $\chi_m(\underline{x}_k)$ and summing overk. Because $\psi_m(\underline{x}_k)$ does not depend on p, q, or l, $\chi_m(\underline{x}_k)$ and the k-summation can be brought inside the p, q, and l summations. Since $\chi_m(\underline{x}_k)$ and $\psi_l(\underline{x}_k)$ are orthogonal, the resulting k-sum is zero. The extension to the general case follows trivially from a repetition of the reasoning in section 5.

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References

Baer, F. and Tribbia, 1977: On complete filtering of gravity modes through nonlinear initialization. Mon. Wea. Rev., 105, 1536-1539.

Ballish, B., 1980: Initialization, theory and application to the NMC Spectral Model. Ph.D. Thesis, Dept. of Meteorology, Univ. of Maryland, 151 pp.

Daley, R., 1978: Variational non-linear normal mode initialization. Tellus, 30, 201-218.

Eliassen, A., 1954: Provisional report on calculation of spatial covariance and autocorrelation of the pressure field. Publs. and Reports, July 1, 1953-June 30, 1954. Inst. for Weather and Climate Res., Oslo, 11 pp.

Leith, C., 1980: Nonlinear normal mode initialization and quasi-geostrophic theory. J. Atmos. Sci., 37, 958-968.

Lorenc, A., 1981: A global three-dimensional multivariation statistical interpolation scheme. Mon. Wea. Rev., 109, 701-721.

Machenhauer, B., 1976: An initialization procedure based on the elimination of gravity oscillations in a spectral, barotropic primitive equation model. Simulation of Large-Scale Processes, Annalen des Meteorlogie, 11, 135-138.

Phillips, N., 1982: Variational analysis and the slow manifold. Mon. Wea. Rev., 110, ________.

Sasaki, Y., 1958: An objective analysis based on the variational method. J. Meteor. Soc. Japan, 36, 77-78.

Temperton, C., 1980: Design of the ECMWF normal mode initialization procedure. In Seminars 1980, Data Assimilation Methods. European Centre for Medium Range Weather Forecasts, Shinfield Park, 454 pp.